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Sums of generalized weighted composition operators from weighted Bergman spaces to weighted Banach spaces

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ABSTRACT: The present research provides both necessary and sufficient conditions for the sum operator $\mathcal{S}_{\mu,\eta}^{k}$ to exhibit boundedness and compactness when mapping from the weighted Bergman spaces $\mathcal{A}_{v}^{p}$ to the weighted Banach spaces $H_{w}^{\infty}(H_{w}^{0})$. This unification encompasses the product of multiplication, differentiation, and composition operators. Furthermore, we provide an example to demonstrate that the boundedness of the operator $\mathcal{S}_{\mu,\eta}^{k}: \mathcal{A}_{v}^{p} \longrightarrow H_{w}^{\infty}$ does not necessarily imply the boundedness of the operator $\mathcal{S}_{\mu,\eta}^{k}: \mathcal{A}_{v}^{p} \longrightarrow H_{w}^{0}$. Also, we present an example of a bounded operator $\mathcal{S}_{\mu,\eta}^{k}: H_{v}^{\infty} \longrightarrow H_{w}^{\infty}$, while the operator $\mathcal{S}_{\mu,\eta}^{k}: \mathcal{A}_{v}^{p} \longrightarrow H_{w}^{\infty}$ is not bounded.

1. INTRODUCTION

The interaction of the theory of composition operators and weighted composition operators with differentiation operators on various analytic function spaces gave rise to a new class of operators known as generalized weighted composition operators and Stević-Sharma type operators, which have been studied recently by many mathematicians [1, 2, 4, 14, 18].

Consider $\mathbb{D}$ as the unit disc in the complex plane $\mathbb{C}$, and let $\mathcal{H}(\mathbb{D})$ denote the space of functions that are analytic on $\mathbb{D}$. Furthermore, let $\Lambda(\mathbb{D})$ represent the set of self-maps that are analytic on $\mathbb{D}$. In a study conducted by Wang et al. [15], the following operator was introduced:

$$\mathcal{S}_{\mu,\eta}^{k} g = \sum_{j=0}^{k} \mu_j \cdot g^{(j)} \circ \eta \quad , g \in \mathcal{H}(\mathbb{D}), \qquad (1)$$

where $k \in \mathbb{N}_0$, $\eta \in \Lambda(\mathbb{D})$ and $\mu = (\mu_j)_{j=0}^{k}$; $\mu_j \in \mathcal{H}(\mathbb{D})$.

Consider a bounded and continuous function $v: \mathbb{D} \to (0, \infty)$, which is commonly referred to as a weight.

The weighted and little weighted spaces of analytic functions are defined as follows:

$$H_v^{\infty} = \left\{ h \in \mathcal{H}(\mathbb{D}): \|h\|_v = \sup_{\zeta \in \mathbb{D}} v(\zeta)|h(\zeta)| < \infty \right\}$$

and

$$H_v^{0} = \left\{ h \in \mathcal{H}(\mathbb{D}): \lim_{|\zeta| \to 1} v(\zeta)|h(\zeta)| = 0 \right\}.$$

Obviously, the space H_v^{∞}, equipped with the norm $\|h\|_v = \sup_{\zeta \in \mathbb{D}} v(\zeta)|h(\zeta)|$, is a Banach space.

Convergence in the norm in H_v^{∞} corresponds to uniform convergence on compact subsets of $\mathbb{D}$. Furthermore, it is evident that H_v^{0} is a closed subspace of H_v^∞. In the special case where $v(\zeta) = 1$, we have $H_v^\infty = H^\infty$.

The weight $\tilde{v}$ associated with v is is defined as follows:

$$\tilde{v}(\zeta) = (\sup\{|h(\zeta)|: \zeta \in H_v^\infty,\ \|\zeta\|_v \leq 1\})^{-1}. \quad (2)$$

A weight v is said to be radial if it satisfies $v(\zeta) = v(|\zeta|)$ for all $\zeta \in \mathbb{D}$. In the work of [5], it has been shown that:

$$\| h \|_v \leq 1 \text{ if and only if } \| h \|_{\tilde{v}} \leq 1; \quad (3)$$

$\tilde{v} \geq v > 0$ and $\tilde{v}$ is continuous and bounded;(4)

for all $\zeta \in \mathbb{D}$, there is f_ζ in B_v^∞, which is the closed unit ball in H_v^∞, such that

$$|h_\zeta(\zeta)| = \frac{1}{\tilde{v}(\zeta)}. \quad (5)$$

This article places significant importance on the condition (L1) introduced by Lusky [10].

$$\inf_{n\in\mathbb{N}} \frac{v(1-2^{-n-1})}{v(1-2^{-n})} > 0. \quad \text{(L1)}$$

By an essential weight v, we mean that there exist a constant $c > 0$ with $v(\zeta) \leq \tilde{v}(\zeta) \leq cv(\zeta)$ for each $\zeta \in \mathbb{D}$. From [6], we know that any radial weight satisfying condition (L1) is also essential. For instance, the weights $v_\alpha(\zeta) = (1-|\zeta|^2)^\alpha, \alpha > 0$, $v_{\log}(\zeta) = (1-|\zeta|)\log\frac{3}{1-|\zeta|}$ and $w_\beta(\zeta) = \left(1-\log(1-|\zeta|^2)\right)^\beta, \beta < 0$ are essential (see [10]). Weighted spaces of analytic functions arise organically when investigating growth conditions of analytic functions. They possess significant applications in various fields, including functional analysis, complex analysis, convolution equations, partial differential equations and distribution theory.

Next, The weighted Bergman space is a class of analytic functions defined in the following manner:

$$\mathcal{A}_v^p = \left\{h \in \mathcal{H}(\mathbb{D}): \|h\|_{v,p} = \left(\int_{\mathbb{D}} v(\zeta)|h(\zeta)|^p dA(\zeta)\right)^{\frac{1}{p}} < \infty\right\}; p \in [0,\infty),$$

$dA(\zeta) = dxdy/\pi$ represent the normalized area measure. The Bergman space A_v^p is a Banach space of analytic functions on $\mathbb{D}$ with the norm $\| h \|_{v,p}$. Based on the references [3, 9, 11], it has been established that when the weight in the weighted Bergman space is radial, the set of polynomials forms a dense subset of the space. In the case where the weight function is defined as $v(\zeta) = 1$, the resulting space is commonly referred to as the classical Bergman space, denoted as $\mathcal{A}_v^p = A_p$. If $v(\zeta) = v_\alpha(\zeta) = (1-|\zeta|^2)^\alpha$, $\alpha > 0$, then $\mathcal{A}_v^p = A_{\alpha,p}$. To delve deeper into Bergman spaces, we recommend [8, 17].

Additionally, we also take into consideration the weight v, which is defined as

$$v(\zeta) = \mathcal{V}(|\zeta|^2) \quad \text{for each } z \in \mathbb{D}, \quad (6)$$

The weight function $\mathcal{V}$ is an analytic function defined on the unit disk $\mathbb{D}$. It satisfies the properties of being non-vanishing, strictly positive on the interval $[0,1)$, and approaches zero as the limit of $\mathcal{V}(r)$ as r approaches 1. These properties can be exemplified through various examples, as demonstrated in [16].

1. If $\mathcal{V}_\alpha(\zeta) = (1-\zeta)^\alpha$, where $\alpha \geq 1$, then $v_\alpha(\zeta) = (1-|\zeta|^2)^\alpha$.

2. If $\mathcal{V}_\alpha(\zeta) = \exp^{-\frac{1}{(1-\zeta)^\alpha}}$, where $\alpha \geq 1$, then $v_\alpha(\zeta) = \exp^{-\frac{1}{(1-|\zeta|^2)^\alpha}}$.

3. If $\mathcal{V}_{\log}^\beta(\zeta) = \left(1-\log(1-\zeta)\right)^\beta, \beta < 0$, then $v_{\log}^\beta(\zeta) = \left(1-\log(1-|\zeta|^2)\right)^\beta$.

4. If $\mathcal{V}(\zeta) = \sin(1-\zeta)$, then $v(z) = \sin(1-|\zeta|^2)$.

Let $a \in \mathbb{D}$. Then we define the function $v_a(\zeta) = \mathcal{V}(\overline{a}\zeta)$ and $\eta_a(\zeta) = \frac{a-\zeta}{1-\overline{a}\zeta}$ for every $\zeta \in \mathbb{D}$. Clearly, v_a is analytic, $\eta_a(\eta_a(\zeta)) = \zeta$ and $\eta_a'(\zeta) = -\frac{1-|a|^2}{(1-\overline{a}\zeta)^2}, \zeta \in \mathbb{D}$. The map η_a which interchanges a and 0 is called Möbius transformation. For non-negative quantities K and M, we denote $K \asymp M$, indicating that $K \preceq M$ and $M \preceq K$, where $K \preceq M$ implies the existence of a positive constant C such that $K \leq CM$.

2. BOUNDEDNESS OF $\mathcal{S}^k_{\mu,\eta}: \mathcal{A}^p_v \to H^\infty_w(H^0_w)$

To establish the main results concerning the operators $\mathcal{S}^k_{\mu,\eta}$, it is necessary to introduce the following lemma, which has been proven in[1].

Lemma 2.1 *Consider a radial weight v, defined as shown in (6), which possesses the following property:*

$$\sup_{a\in\mathbb{D}}\sup_{z\in\mathbb{D}}\frac{v(z)|v_a(\eta_a(z))|}{v(\eta_a(z))} \le C < \infty.$$

Additionally, Suppose the weight function v satisfies condition (L1). In that case, there exists a positive constant C_v such that for any $f \in \mathcal{A}^p_v$,

$$\left|f^{(n)}(z)\right| \le \frac{C_v\|f\|_{v,p}}{(1-|z|^2)^{n+\frac{2}{p}}v(z)^{\frac{1}{p}}}$$

holds for each $z \in \mathbb{D}$ and $n \in \mathbb{N}_0$.

We present the following theorem that provides a characterization the self map $\eta \in \Lambda(\mathbb{D})$ and $\mu = (\mu_j)^k_{j=0}$, $\mu_j \in \mathcal{H}(\mathbb{D})$ which induce bounded operator $\mathcal{S}^k_{\mu,\eta}: \mathcal{A}^p_v \longrightarrow H^\infty_w$.

Theorem 2.2 *Let v be a weight function defined as in Lemma 2.1, and let w be an arbitrary weight function. Suppose $\mu = (\mu_j)^k_{j=0}$, where $\mu_j \in \mathcal{H}(\mathbb{D})$ and $\eta \in \Lambda(\mathbb{D})$. The conditions necessary and sufficient for the boundedness of the operator $\mathcal{S}^k_{\mu,\eta}: \mathcal{A}^p_v \longrightarrow H^\infty_w$ are given by*

$$M_j = \sup_{z\in\mathbb{D}}\frac{w(z)|\mu_j(z)|}{(1-|\eta(z)|^2)^{j+\frac{2}{p}}v(\eta(z))^{\frac{1}{p}}} < \infty, \quad j = 0,\dots,k. \tag{7}$$

Moreover, for the bounded operator $\mathcal{S}^k_{\mu,\eta}: \mathcal{A}^p_v \longrightarrow H^\infty_w$ we have

$$\left\|\mathcal{S}^k_{\mu,\eta}\right\|_{\mathcal{A}^p_v\to H^\infty_w} \asymp \sum_{j=0}^k M_j \asymp \max\{M_j:\ j = 0,1,\dots,k\}. \tag{8}$$

Proof. First let $\mathcal{S}^k_{\mu,\eta}: \mathcal{A}^p_v \longrightarrow H^\infty_w$ be a bounded operator and Let $u = v^{\frac{1}{p}}$ and define $\tilde{u}$ as in (2). It is evident that if v is a radial weight satisfying condition (L1), then u also satisfies condition (L1). Moreover, since u satisfies condition (L1), it is essential. Consequently, for every $z \in \mathbb{D}$, there exists a constant $\gamma > 0$ such that $u(z) \le \tilde{u}(z) \le \gamma u(z)$. Fix $a \in \mathbb{D}$. Based on (5), there is a function $f_{\eta(a)} \in B^\infty_u \subseteq H^\infty_u$ such that

$$\left|f_{\eta(a)}(\eta(a))\right| = \frac{1}{\tilde{u}(\eta(a))} \asymp \frac{1}{u(\eta(a))} = \frac{1}{v(\eta(a))^{\frac{1}{p}}} \tag{9}$$

and so

$$\left|f_{\eta(a)}(\eta(a))\right|^p \asymp \frac{1}{v(\eta(a))}.$$

To prove the condition (7) for $j = k$, if we establish $L_{\eta(a)}(z) = \eta^k_{\eta(a)}(z)f_{\eta(a)}(z)\left(\eta'_{\eta(a)}(z)\right)^{\frac{2}{p}}$, $z \in \mathbb{D}$, then clearly

$$\left\|L_{\eta(a)}\right\|^p_{v,p} \le \sup_{z\in\mathbb{D}} v(z)\left|f_{\eta(a)}(z)\right|^p \int_{\mathbb{D}} \left|\eta_{\eta(a)}(z)\right|^{kp}\left|\eta'_{\eta(a)}(z)\right|^2 dA(z) \le 1.$$

Thus $L_{\eta(a)} \in \mathcal{A}^p_v$, $L^{(j)}_{\eta(a)}(\eta(a)) = 0$ for each $j = 0,1,\dots,k-1$. Also,

$$\left|L^{(k)}_{\eta(a)}(\eta(a))\right| \asymp \frac{k!}{(1-|\eta(a)|^2)^{k+\frac{2}{p}}v(\eta(a))^{\frac{1}{p}}}. \tag{10}$$

Thus, using (10), we get

$$\begin{aligned}\left\|\mathcal{S}^k_{\mu,\eta}\right\|_{\mathcal{A}^p_v\to H^\infty_w} &\ge \left\|\mathcal{S}^k_{\mu,\eta}L_{\eta(a)}\right\|_w \\ &\ge w(a)\left|(\mathcal{S}^k_{\mu,\eta}L_{\eta(a)})(a)\right| \\ &\ge \sum_{j=0}^k w(a)\left|\mu_j(a)L^{(j)}_{\eta(a)}(\eta(a))\right| \\ &\ge \frac{k!w(a)|\mu_k(a)|}{\gamma(1-|\eta(a)|^2)^{k+\frac{2}{p}}v(\eta(a))^{\frac{1}{p}}} \\ &\ge \frac{w(a)|\mu_k(a)|}{\gamma(1-|\eta(a)|^2)^{k+\frac{2}{p}}v(\eta(a))^{\frac{1}{p}}},\end{aligned}$$

which implies that

$$\gamma\left\|\mathcal{S}^k_{\mu,\eta}\right\|_{\mathcal{A}^p_v\to H^\infty_w} \ge \sup_{a\in\mathbb{D}}\frac{w(a)|\mu_k(a)|}{(1-|\eta(a)|^2)^{k+\frac{2}{p}}v(\eta(a))^{\frac{1}{p}}} = M_k. \tag{11}$$

Thus we have established the condition (7) for $j = k$. Now we shall prove the condition (7) for any $0 \le j \le k$. For this, we assume the following inequality

$M_i \leq \gamma(1+\gamma C_v)^{k-i}\left\|\mathcal{S}_{\mu,\eta}^k\right\|_{\mathcal{A}_v^p \to H_w^\infty}$ for each $j+1 \leq i \leq k,$ (12)

and establish it for $i=j$. So, if we define define

$$G_{\eta(a)}(z)=\eta_{\eta(a)}^j(z) f_{\eta(a)}(z)\left(\eta'_{\eta(a)}(z)\right)^{\frac{2}{p}}, z \in \mathbb{D}.$$

Clearly $\left\|G_{\eta(a)}\right\|_{v,p} \leq 1$, $G_{\eta(a)}^{(i)}(\eta(a))=0$ for all $0 \leq i \leq j-1$ and

$$\left|G_{\eta(a)}^{(j)}(\eta(a))\right| \asymp \frac{j!}{(1-|\eta(a)|^2)^{j+\frac{2}{p}} v(\eta(a))^{\frac{1}{p}}}. \quad (13)$$

Further, by applying Lemma 2.1 and (13), it can be easily seen that

$$\begin{aligned}\left\|\mathcal{S}_{\mu,\eta}^k\right\|_{\mathcal{A}_v^p \to H_w^\infty} &\geq \left\|\mathcal{S}_{\mu,\eta}^k G_{\eta(a)}\right\|_w \\ &\geq w(a)\left|(\mathcal{S}_{\mu,\eta}^k G_{\eta(a)})(a)\right| \\ &\geq \sum_{i=0}^k w(a)\left|\mu_i(a) h_{\eta(a)}^{(i)}(\eta(a))\right| \\ &\geq \frac{j!\,w(a)|\mu_j(a)|}{\gamma(1-|\eta(a)|^2)^{j+\frac{2}{p}} v(\eta(a))^{\frac{1}{p}}} - C_v\left\|G_{\eta(a)}\right\|_{v,p} \sum_{i=j+1}^k \frac{w(a)|\mu_i(a)|}{(1-|\eta(a)|^2)^{i+\frac{2}{p}} v(\eta(a))^{\frac{1}{p}}}. \quad (14)\end{aligned}$$

Thus it readily follows from (12) and (13) that

$$\begin{aligned}M_j &= \sup_{a\in\mathbb{D}} \frac{w(a)|\mu_j(a)|}{(1-|\eta(a)|^2)^{j+\frac{2}{p}} v(\eta(a))^{\frac{1}{p}}} \\ &\leq \gamma\left(\left\|\mathcal{S}_{\mu,\eta}^k\right\|_{\mathcal{A}_v^p \to H_w^\infty} + C_v \| G_{\eta(a)} \|_{v,p} \sum_{i=j+1}^k \frac{w(a)|\mu_i(a)|}{(1-|\eta(a)|^2)^{1+\frac{2}{p}} v(\eta(a))^{\frac{1}{p}}}\right) \\ &\leq \gamma\left(1+C_v \sum_{i=j+1}^k \gamma(1+\gamma C_v)^{k-i}\right)\left\|\mathcal{S}_{\mu,\eta}^k\right\|_{\mathcal{A}_v^p \to H_w^\infty} \\ &= \gamma(1+\gamma C_v)^{k-j}\left\|\mathcal{S}_{\mu,\eta}^k\right\|_{\mathcal{A}_v^p \to H_w^\infty} < \infty, \quad 0 \leq j \leq k. \quad (15)\end{aligned}$$

Hence

$$\sum_{j=0}^k M_j \leq \sum_{j=0}^k \gamma(1+\gamma C_v)^{k-j}\left\|\mathcal{S}_{\mu,\eta}^k\right\|_{\mathcal{A}_v^p \to H_w^\infty}. \quad (16)$$

This proves condition (7).

Conversely, let us assume that condition (7) is satisfied. Consider $h \in \mathcal{A}_v^p$. By utilizing Lemma 2.1, we obtain

$$\begin{aligned}\left\|\mathcal{S}_{\mu,\eta}^k h\right\|_w &= \sup_{z\in\mathbb{D}} w(z)\left|\sum_{j=0}^k \mu_j(z) h^{(j)}(\eta(z))\right| \\ &\leq \sup_{z\in\mathbb{D}} w(z) \sum_{j=0}^k \left|\mu_j(z) h^{(j)}(\eta(z))\right| \\ &\leq \sum_{j=0}^k \sup_{z\in\mathbb{D}} \frac{C_v\|h\|_{v,p} w(z)|\mu_j(z)|}{(1-|\eta(z)|^2)^{j+\frac{2}{p}} v(\eta(z))^{\frac{1}{p}}} \\ &\leq C_v \| h \|_{v,p} \sum_{j=0}^k M_j. \quad (17)\end{aligned}$$

Thus,

$$\left\|\mathcal{S}_{\mu,\eta}^k\right\|_{\mathcal{A}_v^p \to H_w^\infty} \leq C_v \sum_{j=0}^k M_j. \quad (18)$$

Hence, it is established that the operator $\mathcal{S}_{\mu,\eta}^k: \mathcal{A}_v^p \to H_w^\infty$ is bounded. Additionally, we have the inequality

$$\max\{M_j: 0 \leq j \leq k\} \leq \sum_{j=0}^k M_j \leq (k+1)\max\{M_j: 0 \leq j \leq k\}. \quad (19)$$

From (18), (16), and (19), it is evident that the asymptotic relation (8) follows.

Corollary 2.3 *Let v be a weight function defined as in Lemma 2.1, and let w be an arbitrary weight function. Suppose $\mu=(\mu_j)_{j=0}^k$, where $\mu_j \in \mathcal{H}(\mathbb{D})$ and $\eta \in \Lambda(\mathbb{D})$. Then $\mathcal{D}_{\mu_j,\eta}^j: \mathcal{A}_v^p \longrightarrow H_w^\infty$ is bounded if and only if (7) is satisfied for every j, $0 \leq j \leq k$.*

Proof. If the condition (7) holds, then by using the same technique of Theorem 2.2, it can be easily proved that the operator $\mathcal{D}_{\mu_j,\eta}^j$ is bounded. Also, if the operator $\mathcal{D}_{\mu_j,\eta}^j$ is bounded, then clearly the operator $\mathcal{S}_{\mu,\eta}^k = \sum_{j=0}^k \mathcal{D}_{\mu_j,\eta}^j$ is bounded and hence the condition (7) follows from Theorem 2.2.

Since $H_w^0 \subseteq H_w^\infty$, the boundedness of $\mathcal{S}_{\mu,\eta}^k: \mathcal{A}_v^p \to H_w^\infty$ does not imply the boundedness of $\mathcal{S}_{\mu,\eta}^k: \mathcal{A}_v^p \to H_w^0$.The subsequent theorem characterizes the boundedness of $\mathcal{S}_{\mu,\eta}^k: \mathcal{A}_v^p \to H_w^0$.

Corollary 2.4 *Let v be a weight function defined as in Lemma 2.1, and let w be an arbitrary weight function. Suppose $\mu = (\mu_j)_{j=0}^{k}$, where $\mu_j \in \mathcal{H}(\mathbb{D})$ and $\eta \in \Lambda(\mathbb{D})$. Then $\mathcal{S}_{\mu,\eta}^{k}: \mathcal{A}_v^p \to H_w^0$ is bounded if and only if $\mathcal{S}_{\mu,\eta}^{k}: \mathcal{A}_v^p \to H_w^\infty$ is bounded, and $\mu_j \in H_w^0$, $j = 0,1,\dots,k$.*

Proof. Since $H_w^0 \subseteq H_w^\infty$, if $\mathcal{S}_{\mu,\eta}^{k}: \mathcal{A}_v^p \to H_w^0$ is bounded, then $\mathcal{S}_{\mu,\eta}^{k}: \mathcal{A}_v^p \to H_w^\infty$ is also bounded. Further, if we define $f_j(z) = z^j$, $0 \le j =\le k$, then we have $f_j \in \mathcal{A}_v^p$. Hence $\mathcal{S}_{\mu,\eta}^{k} f_j \in H_w^0$ implies that $\mu_j \in H_w^0$.

Conversely, Assuming the boundedness of the operator $\mathcal{S}_{\mu,\eta}^{k}: \mathcal{A}_v^p \to H_w^\infty$ and considering $\mu_j \in H_w^0$, $j = 0,1,\dots,k$. Let $f \in \mathcal{A}_v^p$. If we take $p(z)$ as polynomial, then we have $\lim_{|z|\to 1} w(z)|\mathcal{S}_{\mu,\eta}^{k} p(z)| = \lim_{|z|\to 1} w(z)\left|\sum_{j=0}^{k} \mu_j(z) p^{(j)}(\eta(z))\right| \le \sum_{j=0}^{k} \lim_{|z|\to 1} w(z)\left|\mu_j(z)\right|\left\|p^{(j)}\right\|_\infty = 0.$

Thus, $\mathcal{S}_{\mu,\eta}^{k} p \in H_w^0$. Since it is a well-known fact that the set of polynomials is dense in $\mathcal{A}_v^p$ for the radial weight v (refer to [3, p.10], [9, p.343], or [11, p.134]), we can find a sequence of polynomials $\{p_n\}_{n\in\mathbb{N}}$ such that $\|f - p_n\|_{v,p} \to 0$ as $n \to \infty$. Hence,

$$\left\|\mathcal{S}_{\mu,\eta}^{k} f - \mathcal{S}_{\mu,\eta}^{k} p_n\right\|_w \le \left\|\mathcal{S}_{\mu,\eta}^{k}\right\|_{\mathcal{A}_v^p \to H_w^\infty} \|f - p_n\|_{v,p} \to 0 \ as \ n \to \infty.$$

As H_w^0 is a closed subspace of H_w^∞, we have $\mathcal{S}_{\mu,\eta}^{k}(\mathcal{A}_v^p) \subseteq H_w^0$. Consequently, $\mathcal{S}_{\mu,\eta}^{k}: \mathcal{A}_v^p \to H_w^0$ is bounded.

Remark 2.5 *By referring to Theorem 2.2, Corollary 2.3, and Corollary 2.4, it becomes evident that the operator $\mathcal{D}_{\mu_j,\eta}^{j}: \mathcal{A}_v^p \longrightarrow H_w^0$ is bounded if and only if $\mathcal{D}_{\mu_j,\eta}^{j}: \mathcal{A}_v^p \longrightarrow H_w^\infty$ is bounded, and $\mu_j \in H_w^0$, $j = 0,1,\dots,k$.*

Based on Corollary 2.4, it can be deduced that if the operator $\mathcal{S}_{\mu,\eta}^{k}: \mathcal{A}_v^p \to H_w^0$ is bounded, then $\mathcal{S}_{\mu,\eta}^{k}: \mathcal{A}_v^p \to H_w^\infty$ is also bounded. However, it is important to note that the converse may not hold true. To illustrate this, we provide the following example:

Example 2.6 *Consider $p = 1$, $v(z) = 1 - |z|^2$ and $w(z) = \left(1 - \frac{|z|^2}{4}\right)^4$. Define $\eta(z) = \frac{z}{2}$. Let $\mu = (\mu_j)_{j=0}^{k}$, where $\mu_0(z) = e^z$, $\mu_1(z) = e^{z^2}$ and $\mu_i = 0$ for each $i = 2,3,\dots,k$. Then we have*

$$\frac{w(z)|\mu_0(z)|}{(1 - |\eta(z)|^2)^{\frac{2}{p}} v(\eta(z))^{\frac{1}{p}}} \le \left(1 - \frac{|z|^2}{4}\right) e^{|z|} < e \ for\ each\ z \in \mathbb{D},$$

$$\frac{w(z)|\mu_1(z)|}{(1 - |\eta(z)|^2)^{1+\frac{2}{p}} v(\eta(z))^{\frac{1}{p}}} = \frac{\left(1 - \frac{|z|^2}{4}\right)^4 |e^{z^2}|}{\left(1 - \frac{|z|^2}{4}\right)^3 \left(1 - \frac{|z|^2}{4}\right)} < e^{|z|^2} < e \quad \text{for each } z \in \mathbb{D}.$$

and

$$\frac{w(z)|\mu_i(z)|}{(1 - |\eta(z)|^2)^{i+\frac{2}{p}} v(\eta(z))^{\frac{1}{p}}} = 0 \quad \text{for each } i = 2,3,\dots,k.$$

Thus, the condition of Theorem2.2 is satisfied and hence $\mathcal{S}_{\mu,\eta}^{k}: \mathcal{A}_v^p \to H_w^\infty$ is bounded. However, for $z = r$, we observe that

$$\lim_{|z|\to 1} w(z)|\mu_0(z)| = \lim_{r\to 1} \left(1 - \frac{r^2}{4}\right)^4 e^r = \left(\frac{3}{4}\right)^4 e \ne 0.$$

That is, $\mu_0 \notin H_w^0$. Thus, according to Corollary2.4, $\mathcal{S}_{\mu,\eta}^{k}: \mathcal{A}_v^p \to H_w^0$ is not bounded.

It is evident that if the operato $\mathcal{S}_{\mu,\eta}^{k}: \mathcal{A}_v^p \to H_w^\infty$ is bounded, then $\mathcal{S}_{\mu,\eta}^{k}: H_v^\infty \to H_w^\infty$ is also bounded. However, it is important to note that the converse may not hold true, as demonstrated in the following example:

Example 2.7 *Consider $p = 1$, $v(z) = 1 - |z|^2$ and $w(z) = (1 - |z|)^4$. Let $\eta(z) = \frac{z+1}{2}$ and $\mu = (\mu_j)_{j=0}^{k}$, where $\mu_j(z) = \frac{1}{(1-z)^j}$, $i = 2,3,\dots,k$. We have*

$$\frac{w(z)|\mu_0(z)|}{v(\eta(z))} = \frac{(1-|z|)^4 2|z|}{(1-|\frac{z+1}{2}|^2)} \leq \frac{2(1-|z|)^4}{\left(\frac{1-|z|}{2}\right)\left(\frac{|z|+1}{2}\right)} \leq 8(1-|z|)^3 < \infty, \quad (20)$$

$$\frac{w(z)|\mu_1(z)|}{(1-|\eta(z)|^2)v(\eta(z))} = \frac{(1-|z|)^4 \frac{1}{|1-z|^2}}{\left(1-|\frac{z+1}{2}|^2\right)^2} \leq \frac{(1-|z|)^2}{\left(1-|\frac{z+1}{2}|^2\right)^2} \leq \frac{(1-|z|)^2}{\left(\frac{1-|z|}{2}\right)^2\left(\frac{|z|+1}{2}\right)^2} \leq 16 < \infty. \quad (21)$$

By examining (20) and (21), it becomes evident that the operator $\mathcal{S}_{\mu,\eta}^k: H_v^\infty \to H_w^\infty$ is bounded. Looking at it from a different angle, we can consider

$$\frac{w(z)|\mu_1(z)|}{(1-|\eta(z)|^2)^{1+\frac{2}{p}} v(\eta(z))^{\frac{1}{p}}} = \frac{(1-|z|)^4 \frac{1}{|1-z|^2}}{\left(1-|\frac{z+1}{2}|^2\right)^4}.$$

For $z = r$,

$$\frac{w(r)|\mu_1(r)|}{(1-|\eta(r)|^2)^3 v(\eta(r))} = \frac{(1-r)^2}{\left(1-\left(\frac{r+1}{2}\right)^2\right)^4} \to \infty \quad \text{as } r \to 1.$$

Therefore, we can conclude that the operator $\mathcal{S}_{\mu,\eta}^k: \mathcal{A}_v^p \to H_w^\infty$ is unbounded.

3. COMPACTNESS OF $\mathcal{S}_{\mu,\eta}^k: H_v \to H_w^\infty(H_w^0)$

In order to characterize the self map $\eta \in \Lambda(\mathbb{D})$ and $\mu = (\mu_j)_{j=0}^k$, which induce compact operator $\mathcal{S}_{\mu,\eta}^k$, we need the following result and the proof can be deduced from Proposition 3.11 [7].

Lemma 3.1 *Consider $\mu = (\mu_j)_{j=0}^k$ be such that $\mu_j \in \mathcal{H}(\mathbb{D})$ and $\eta \in \Lambda(\mathbb{D})$. Then $\mathcal{S}_{\mu,\eta}^k: \mathcal{A}_v^p \longrightarrow H_w^\infty$ is compact if and only if $\mathcal{S}_{\mu,\eta}^k: \mathcal{A}_v^p \longrightarrow H_w^\infty$ is bounded and for any bounded sequence $\{f_n\}_{n\in\mathbb{N}}$ in $\mathcal{A}_v^p$ such that $f_n \to 0$ uniformly on compact subsets of $\mathbb{D}$ as $n \to \infty$, $\|\mathcal{S}_{\mu,\eta}^k f_n\|_w \to 0$ as $n \to \infty$.*

Theorem 3.2 *Let v be a weight function defined as in Lemma 2.1, and let w be an arbitrary weight function. Suppose $\mu = (\mu_j)_{j=0}^k$, where $\mu_j \in \mathcal{H}(\mathbb{D})$ and $\eta \in \Lambda(\mathbb{D})$. The conditions necessary and sufficient for the compactness of the operator $\mathcal{S}_{\mu,\eta}^k: \mathcal{A}_v^p \longrightarrow H_w^\infty$ are given by*

$$\lim_{r\to 1} \sup_{|\eta(z)|>r} \frac{w(z)|\mu_j(z)|}{(1-|\eta(z)|^2)^{j+\frac{2}{p}} v(\eta(z))^{\frac{1}{p}}} = 0, \ j = 0,1,\dots,k. \quad (22)$$

Proof. First, we assume that condition (22) holds and $\mathcal{S}_{\mu,\eta}^k: \mathcal{A}_v^p \to H_w^\infty$ is bounded. By considering the function $f_j(z) = z^j, \ z \in \mathbb{D}$ in $\mathcal{A}_v^p$, we can have

$$K_j = \sup_{z\in\mathbb{D}} w(z)|\mu_j(z)| < \infty, \ 0 \leq j \leq k. \quad (23)$$

Now let $\{f_n\}_{n\in\mathbb{N}}$ be a bounded sequence in $\mathcal{A}_v^p$ such that it converges to zero uniformly on compact subsets of $\mathbb{D}$. To show that $\mathcal{S}_{\mu,\eta}^k$ is compact, in view of Lemma 3.1, it is enough to show that $\|\mathcal{S}_{\mu,\eta}^k f_n\|_w \to 0$ as $n \to \infty$. Based on condition (22), we can conclude that for any $\epsilon > 0$, there exists $r \in (0,1)$ such that whenever $r < |\eta(z)| < 1$, the following inequality holds:

$$\frac{w(z)|\mu_j(z)|}{(1-|\eta(z)|^2)^{j+\frac{2}{p}} v(\eta(z))^{\frac{1}{p}}} < \epsilon, \ 0 \leq j \leq k. \quad (24)$$

Since the sequence $\{f_n\}_{n\in\mathbb{N}}$ converges to zero uniformly on compact subsets of $\mathbb{D}$, we can apply Cauchy's estimates to conclude that $\left\{f_n^{(j)}\right\}_{n\in\mathbb{N}}$, $j = 0,1,\dots,k$ also converges to zero uniformly on compact subsets of $\mathbb{D}$. Therefore, there exists $n_0 \in \mathbb{N}$ such that, for $|\eta(z)| \leq r$ and $n \geq n_0$, the following holds:

$$\left|f_n^{(j)}(\eta(z))\right| < \epsilon, \ j = 0,1,\dots,k. \quad (25)$$

Using (23) and (25), we have

$$\begin{aligned}\sup_{|\eta(z)|\leq r} w(z)\left|\mu_j(z) f_n^{(j)}(\eta(z))\right| &\leq \epsilon \sup_{|\eta(z)|\leq r} w(z)\left|\mu_j(z)\right| \\ &\leq \epsilon K_j, \ j = 0,1,\dots,k. \quad (26)\end{aligned}$$

As the sequence $\{f_n\}_{n\in\mathbb{N}}$ is bounded in $\mathcal{A}_v^p$, we have $\sup_n \|f_n\|_{v,p} \leq M$. Consequently, combining (24), (26), and Lemma 2.1, we can deduce that:

$$\|\mathcal{S}_{\mu,\eta}^k f_n\|_w = \sup_{z\in\mathbb{D}} w(z) \left|\sum_{j=0}^k \mu_j(z) f_n^{(j)}(\eta(z))\right|$$

$$= \max\left\{ \sup_{r<|\eta(z)|<1} w(z)\left|\sum_{j=0}^{k} \mu_j(z) f_n^{(j)}(\eta(z))\right|, \right.$$

$$\left. \sup_{|\eta(z)|\le r} w(z)\left|\sum_{j=0}^{k} \mu_j(z) f_n^{(j)}(\eta(z))\right| \right\}$$

$$\le \sum_{j=0}^{k} \sup_{r<|\eta(z)|<1} w(z)\left|\mu_j(z) f_n^{(j)}(\eta(z))\right|$$

$$+ \sum_{j=0}^{k} \sup_{|\eta(z)|\le r} w(z)\left|\mu_j(z) f_n^{(j)}(\eta(z))\right|$$

$$\le \sum_{j=0}^{k} \sup_{r<|\eta(z)|<1} \frac{C_v \|f_n\|_{v,p} w(z)|\mu_j(z)|}{(1-|\eta(z)|^2)^{j+\frac{2}{p}} v(\eta(z))^{\frac{1}{p}}} + \epsilon \sum_{j=0}^{k} K_j$$

$$\le \left((k+1)MC_v + \sum_{j=0}^{k} K_j\right)\epsilon.$$

we can conclude that the operator $\mathcal{S}_{\mu,\eta}^k : \mathcal{A}_v^p \to H_w^\infty$ is compact.

Conversely, assuming that the operator $\mathcal{S}_{\mu,\eta}^k : \mathcal{A}_v^p \to H_w^\infty$ is compact. Clearly $\mathcal{S}_{\mu,\eta}^k : \mathcal{A}_v^p \to H_w^\infty$ is bounded, we can observe that it is also bounded. Now, we aim to establish condition (22). Specifically, for $j = k$, let $\{z_n\}_{n\in\mathbb{N}}$ be a sequence in $\mathbb{D}$ such that $|\eta(z_n)| \to 1$ as $n \to \infty$, satisfying the following: $\lim_{r\to 1} \sup_{|\eta(z)|>r} \frac{w(z)|\mu_k(z)|}{(1-|\eta(z)|^2)^{k+\frac{2}{p}} v(\eta(z))^{\frac{1}{p}}} = \lim_{n\to\infty} \frac{w(z_n)|\mu_k(z_n)|}{(1-|\eta(z_n)|^2)^{k+\frac{2}{p}} v(\eta(z_n))^{\frac{1}{p}}}$.

After selecting a subsequence, we can assume that there exists $n_0 \in \mathbb{N}$ such that $|\eta(z_n)|^n \ge \frac{1}{2}$ for every $n \ge n_0$. Moreover, analogous to (9), we can find $f_{\eta(z_n)} \in B_v^\infty$ satisfying the following expression:

$$\left|f_{\eta(z_n)}(\eta(z_n))\right| \asymp \frac{1}{v(\eta(z_n))^{\frac{1}{p}}}. \tag{27}$$

Let us consider the function for each n as follows:

$$g_n(z) = \eta_{\eta(z_n)}^k(z)\left(\eta'_{\eta(z_n)}(z)\right)^{\frac{2}{p}} f_{\eta(z_n)}(z) z^n, \quad z \in \mathbb{D}. \tag{28}$$

Clearly $g_n \in \mathcal{A}_v^p$ and $\|g_n\|_{v,p} \le 1$. Also, $g_n^{(j)}(\eta(z_n)) = 0, j < k$ and

$$\left|g_n^{(k)}(\eta(z_n))\right| \asymp \frac{k!\,|\eta(z_n)|^n}{(1-|\eta(z_n)|^2)^{k+\frac{2}{p}} v(\eta(z_n))^{\frac{1}{p}}}.$$

As $g_n \to 0$ uniformly on compact subsets of $\mathbb{D}$, we can apply Lemma 3.1 to conclude that $\left\|\mathcal{S}_{\mu,\eta}^k g_n\right\|_w \to 0$ as $n \to \infty$. Therefore, we have the following:

$$\left\|\mathcal{S}_{\mu,\eta}^k g_n\right\|_w \ge w(z_n)\left|\sum_{j=0}^{k} \mu_j(z_n) g_n^{(j)}(\eta(z_n))\right|$$

$$\ge \frac{k!\,w(z_n)|\mu_k(z_n)||\eta(z_n)|^n}{\lambda(1-|\eta(z_n)|^2)^{k+\frac{2}{p}} v(\eta(z_n))^{\frac{1}{p}}}$$

$$\ge \frac{w(z_n)|\mu_k(z_n)|}{2\lambda(1-|\eta(z_n)|^2)^{k+\frac{2}{p}} v(\eta(z_n))^{\frac{1}{p}}}. \tag{29}$$

Based on (29), we can deduce that:

$$\lim_{n\to\infty} \frac{w(z_n)|\mu_k(z_n)|}{(1-|\eta(z_n)|^2)^{k+\frac{2}{p}} v(\eta(z_n))^{\frac{1}{p}}} = 0. \tag{30}$$

This establishes condition (22) for $j = k$. Now, let us consider the case where $0 \le j \le k-1$. Similarly, we assume the existence of a sequence $\{z_n\}_{n\in\mathbb{N}}$ in $\mathbb{D}$ such that $|\eta(z_n)| \to 1$ as $n \to \infty$. We can choose this sequence such that:

$$\lim_{r\to 1} \sup_{|\eta(z)|>r} \frac{w(z)|\mu_i(z)|}{(1-|\eta(z)|^2)^{i+\frac{2}{p}} v(\eta(z))^{\frac{1}{p}}} = \lim_{n\to\infty} \frac{w(z_n)|\mu_i(z_n)|}{(1-|\eta(z_n)|^2)^{i+\frac{2}{p}} v(\eta(z_n))^{\frac{1}{p}}} = 0 \tag{31}$$

for $j+1 \le i \le k$ and we establish (31) for $i = j$. For this, consider:

$$\lim_{r\to 1} \sup_{|\eta(z)|>r} \frac{w(z)|\mu_j(z)|}{(1-|\eta(z)|^2)^{j+\frac{2}{p}} v(\eta(z))^{\frac{1}{p}}} = \lim_{n\to\infty} \frac{w(z_n)|\mu_j(z_n)|}{(1-|\eta(z_n)|^2)^{j+\frac{2}{p}} v(\eta(z_n))^{\frac{1}{p}}}. \tag{32}$$

Again, similar to (28), define: $h_n(z) = \eta_{\eta(z_n)}^j(z)\left(\eta'_{\eta(z_n)}(z)\right)^{\frac{2}{p}} f_{\eta(z_n)}(z) z^n, \quad z \in \mathbb{D}.$

Thus, $\{h_n\}_{n\in\mathbb{N}} \in \mathcal{A}_v^p$ and $\|h_n\|_{v,p} \leq 1$. Also, clearly $h_n^{(i)}(\eta(z_n)) = 0$ for all $0 \leq i \leq j-1$ and

$$\left|h_n^{(j)}(\eta(z_n))\right| \asymp \frac{j!|(\eta(z_n))|^n}{(1-|\eta(z_n)|^2)^{j+\frac{2}{p}}v(\eta(z_n))^{\frac{1}{p}}}. \quad (33)$$

Due to the uniform convergence of $h_n \to 0$ on compact subsets of $\mathbb{D}$, once again employing Lemma 3.1, $\left\|\mathcal{S}_{\mu,\eta}^k h_n\right\|_w \to 0$ as $n \to \infty$. Thus, using (33) and Lemma 2.1, it follows that

$$\left\|\mathcal{S}_{\mu,\eta}^k h_n\right\|_w \geq w(z_n)\left|\mu_j(z_n)h_n^{(j)}(\eta(z_n))\right| - w(z_n)\sum_{i=j+1}^k \left|\mu_i(z_n)h_n^{(i)}(\eta(z_n))\right|$$
$$\geq \frac{j!w(z_n)|\mu_j(z_n)|}{2\lambda(1-|\eta(z_n)|^2)^{j+\frac{2}{p}}v(\eta(z_n))^{\frac{1}{p}}} - \sum_{i=j+1}^k \frac{C_v\|h_n\|_{v,p}w(z_n)|\mu_i(z_n)|}{(1-|\eta(z_n)|^2)^{i+\frac{2}{p}}v(\eta(z_n))^{\frac{1}{p}}}. \quad (34)$$

Further, using (31), (34) implies that

$$\lim_{n\to\infty} \frac{w(z_n)|\mu_j(z_n)|}{(1-|\eta(z_n)|^2)^{j+\frac{2}{p}}v(\eta(z_n))^{\frac{1}{p}}} = 0.$$

The verification of condition (22) establishes its validity, thereby finalizing the proof of the theorem.

Corollary 3.3 *Let v be a weight function defined as in Lemma 2.1, and let w be an arbitrary weight function. Suppose $\mu = (\mu_j)_{j=0}^k$, where $\mu_j \in \mathcal{H}(\mathbb{D})$ and $\eta \in \Lambda(\mathbb{D})$. The conditions necessary and sufficient for the compactness of the operator $\mathcal{D}_{\mu_j,\eta}^j: \mathcal{A}_v^p \to H_w^\infty$ are given by*

$$\lim_{r\to 1}\sup_{|\eta(z)|>r} \frac{w(z)|\mu_j(z)|}{(1-|\eta(z)|^2)^{j+\frac{2}{p}}v(\eta(z))^{\frac{1}{p}}} = 0. \quad (35)$$

Next, we shall utilize the lemma presented in [12] (specifically, Lemma 2.1) to characterize the compactness of the operator $\mathcal{S}_{\mu,\eta}^k: \mathcal{A}_v^p \to H_w^0$.

Lemma 3.4 *Suppose w is an arbitrary weight and K is a closed set in H_w^0. The set K is compact if and only if it is bounded and satisfies the following condition*

$$\lim_{|z|\to 1}\sup_{f\in K} w(z)|f(z)| = 0.$$

Remark 3.5 *When the set K is not closed, the term "compact" in Lemma 3.4 can be substituted with the term "relatively compact."*

Theorem 3.6 *Let v be a weight function defined as in Lemma 2.1, and let w be an arbitrary weight function. Suppose $\mu = (\mu_j)_{j=0}^k$, where $\mu_j \in \mathcal{H}(\mathbb{D})$ and $\eta \in \Lambda(\mathbb{D})$ are given by*

$$\lim_{|z|\to 1} \frac{w(z)|\mu_j(z)|}{(1-|\eta(z)|^2)^{j+\frac{2}{p}}v(\eta(z))^{\frac{1}{p}}} = 0, \ 0 \leq j \leq k. \quad (36)$$

Proof. If condition (36) holds, then clearly, condition of Theorem 2.2 is satisfied. Thus, the operator $\mathcal{S}_{\mu,\eta}^k: \mathcal{A}_v^p \to H_w^\infty$ is bounded. Also, since

$$(1-|\eta(z)|^2)^{j+\frac{2}{p}}v(\eta(z))^{\frac{1}{p}} \leq C_j, \ j = 0,1,\dots,k,$$

from (36), we have

$$\lim_{|z|\to 1} w(z)\left|\mu_j(z)\right| = \lim_{|z|\to 1} \frac{w(z)\left|\mu_j(z)\right|(1-|\eta(z)|^2)^{j+\frac{2}{p}}v(\eta(z))^{\frac{1}{p}}}{(1-|\eta(z)|^2)^{j+\frac{2}{p}}v(\eta(z))^{\frac{1}{p}}}$$
$$\leq \lim_{|z|\to 1} \frac{C_j w(z)\left|\mu_j(z)\right|}{(1-|\eta(z)|^2)^{j+\frac{2}{p}}v(\eta(z))^{\frac{1}{p}}} = 0,$$

$j = 0,1,\dots,k$.

Thus, $\mu_j \in H_w^0$, $0 \leq j \leq k$ and hence $\mathcal{S}_{\mu,\eta}^k: \mathcal{A}_v^p \to H_w^0$ is bounded. Let $f \in \mathcal{A}_v^p$. By utilizing Lemma 2.1, we obtain:

$$w(z)\left|\left(\mathcal{S}_{\mu,\eta}^k f\right)(z)\right| = w(z)\left|\sum_{j=0}^k \mu_j(z)f^{(j)}(\eta(z))\right| \leq \sum_{j=0}^k \frac{C_v\|f\|_{v,p}w(z)|\mu_j(z)|}{(1-|\eta(z)|^2)^{j+\frac{2}{p}}v(\eta(z))^{\frac{1}{p}}}. \quad (37)$$

Consider the sets $S = \left\{f \in \mathcal{A}_v^p: \|f\|_{v,p} \leq 1\right\}$ and $K = \mathcal{S}_{\mu,\eta}^k(S)$. It is evident that K is bounded in H_w^0. Therefore, by utilizing condition (36) in (37), we can conclude that

$$\lim_{|z|\to 1}\sup_{f\in S} w(z)\left|\left(\mathcal{S}_{\mu,\eta}^k f\right)(z)\right| = 0. \quad (38)$$

Therefore, considering Lemma 3.4, we can establish the compactness of the operator. $\mathcal{S}_{\mu,\eta}^k: \mathcal{A}_v^p \to H_w^0$.

Conversely, suppose that the operator $\mathcal{S}^k_{\mu,\eta}: \mathcal{A}^p_v \to H^0_w$ is compact. Since the operator $\mathcal{S}^k_{\mu,\eta}: \mathcal{A}^p_v \to H^0_w$ is bounded, we have already shown that

$$\lim_{|z|\to 1} w(z)\left|\mu_j(z)\right| = 0,\ 0 \le j \le k. \qquad (39)$$

As the operator $\mathcal{S}^k_{\mu,\eta}: \mathcal{A}^p_v \to H^\infty_w$ is compact, we can apply Corollary 2.4 to conclude that

$$\lim_{r\to 1} \sup_{|\eta(z)|>r} \frac{w(z)|\mu_j(z)|}{(1-|\eta(z)|^2)^{j+\frac{2}{p}} v(\eta(z))^{\frac{1}{p}}} = 0,\ j = 0,1,\ldots,k. \qquad (40)$$

To prove (36), fix $0 \le j \le k$ and let $\epsilon > 0$. Then according to (40), there exists $r_j \in (0,1)$ such that whenever $r_j < |\eta(z)| < 1$, we have

$$\frac{w(z)|\mu_j(z)|}{(1-|\eta(z)|^2)^{j+\frac{2}{p}} v(\eta(z))^{\frac{1}{p}}} < \epsilon. \qquad (41)$$

Let

$$E_j = \inf_{|t|\le r_j} (1-|t|^2)^{j+\frac{2}{p}} v(t)^{\frac{1}{p}}. \qquad (42)$$

Then it follows from (42) that

$$E_j \le (1-|\eta(z)|^2)^{j+\frac{2}{p}} v(\eta(z))^{\frac{1}{p}}, \qquad (43)$$

for $|\eta(z)| \le r_j$. Let $\epsilon_j = \epsilon E_j$. Based on (39), we can deduce the existence of $\delta_j \in (0,1)$ such that

$$w(z)|\mu_j(z)| < \epsilon_j \qquad (44)$$

whenever $\delta_j < |z| < 1$. Further, it follows from (43) and (44) that

$$\frac{w(z)|\mu_j(z)|}{(1-|\eta(z)|^2)^{j+\frac{2}{p}} v(\eta(z))^{\frac{1}{p}}} < \epsilon \qquad (45)$$

whenever $|z| > \delta_j$ and $|\eta(z)| \le r_j$. Thus, (41) and (45), implies that

$$\frac{w(z)|\mu_j(z)|}{(1-|\eta(z)|^2)^{j+\frac{2}{p}} v(\eta(z))^{\frac{1}{p}}} < \epsilon \qquad (46)$$

whenever $|z| > \delta_j$. By establishing condition (36), we have successfully completed the proof of the theorem.

Remark 3.7 *From Theorem , Corollary and Theorem, it is clear that The operator $\mathcal{D}^j_{\mu_j,\eta}: \mathcal{A}^p_v \to H^0_w$ is compact if and only if*

$$\lim_{|z|\to 1} \frac{w(z)|\mu_j(z)|}{(1-|\eta(z)|^2)^{j+\frac{2}{p}} v(\eta(z))^{\frac{1}{p}}} = 0$$

4. CONCLUSION

This paper characterize the self map η and $\mu = (\mu_j)^k_{j=0}$ such that $\mu_j \in \mathcal{H}(\mathbb{D})$, which induce bounded and compact operators $\mathcal{S}^k_{\mu,\eta}$ from the weighted Bergman spaces $\mathcal{A}^p_v$ to the weighted Banach spaces $H^\infty_w(H^0_w)$ (Theorem 2.2 and Theorem 3.2). Also, we give an example to show the boundedness of the operator $T^k_{\mu,\eta}: \mathcal{A}^p_v \longrightarrow H^\infty_w$ not necessarily imply the operator $T^k_{\mu,\eta}: \mathcal{A}^p_v \longrightarrow H^0_w$ is bounded.

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